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A Refinement of Kolmogorov's Inequality*

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For any n-times differentiable function f with uniform bounds on f and $f^{(n)}$, we study the pair of values $(f^{(j)}(t), f^{(j+1)}(t))$ for an arbitrary real t and a prescribed $j = 0,..., n-1$. A given value of $f^{(j)}(t)$ determines admissible values for $f^{(j+1)}(t)$. These values are exactly determined in terms of the Euler spline $\mathscr{E}_n(t)$. Special differentiation formulas of cardinal interpolation type are developed to solve the problem.

1. Introduction

In 1939 Kolmogorov [4] proved a sharp inequality between the supremum norms of the successive derivatives of a function. With $n \geqslant 2$ and values for $\|f\|$ and $\|f^{(n)}\|$ he found best possible estimates for $\|f^{(j)}\|$, $1 \leqslant j \leqslant n-1$; here, and in all that follows, the norm is the supremum norm taken over the entire real axis. The inequality is intrinsically tied up with the so called Euler spline function $\mathscr{E}_n(s)$ and can be considered as a characteristic property of $\mathscr{E}_n$. In fact, if we set

$$\gamma_{jn} = \| \mathscr{E}_n^{(j)} \|, \qquad j = 1,..., n,$$

then Kolmogorov's Theorem takes on the following form:

Suppose f has an absolutely continuous $(n-1)$th derivative and satisfies

$$\|f\| \leqslant 1, \qquad \|f^{(n)}\| \leqslant \gamma_{nn}. \tag{1.1}$$

Then also

$$\|f^{(j)}\| \leqslant \gamma_{jn}, \qquad j = 1,..., n-1. \tag{1.2}$$

These inequalities are best possible as they are equalities for $\mathscr{E}_n(s)$.

The constants γ_{jn} can be readily computed from the Fourier series of $\mathscr{E}_n(s)$. By a change of scale in both axes we can always arrange to have (1.1)

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for any given function f. The Euler spline $\mathscr{E}_n$ occurs most naturally within the context of cardinal spline interpolation where it appears as the unique interpolant of the sequence $(-1)^\nu$; we refer the reader to [9] pages 39–40 and also to [8] for background information on these remarkable functions. Most importantly, we need the following three properties of $\mathscr{E}_n(s)$:

(i) $\| \mathscr{E}_n(s) \| = 1$

(ii) $\mathscr{E}_n(\nu) = (-1)^\nu$ for all integers ν;

(iii) if n is even and ν any integer,

$$(-1)^{n/2+\nu} \mathscr{E}_n^{(n)}(s) = \gamma_{nn} \qquad \text{when} \quad \nu - \tfrac{1}{2} < s < \nu + \tfrac{1}{2}\,; \tag{1.3}$$

if n is odd and ν any integer,

$$(-1)^{(n-1)/2+\nu} \mathscr{E}_n^{(n)}(s) = \gamma_{nn} \qquad \text{when} \quad \nu - 1 < s < \nu.$$

These properties characterize $\mathscr{E}_n(s)$ and are sufficient for our purposes.

For convenience, let us denote by $\mathscr{F}_n$ all those functions f satisfying the hypotheses (1.1) of the Theorem. Now for each $j = 0, 1,..., n-1$, define

$$\mathscr{A}_j = \{(f^{(j)}(s), f^{(j+1)}(s))\}$$

where f ranges over the whole class $\mathscr{F}_n$ and s ranges over the whole real axis. Since $\mathscr{F}_n$ is invariant under shifts of origin, we may set $s = 0$ or any prescribed value t if convenient. When we view $\mathscr{A}_j$ as a subset of the $x - y$ plane with

$$x = f^{(j)}(s) \qquad \text{and} \qquad y = f^{(j+1)}(s),$$

several geometric features become immediately obvious. Each $\mathscr{A}_j$ is convex. Also as $f \in \mathscr{F}_n$ implies $\pm f(\pm s) \in \mathscr{F}_n$, we easily establish that $\mathscr{A}_j$ is symmetric in each axis. And from the Kolmogorov Theorem we conclude that $\mathscr{A}_j$ is a bounded set; more precisely, it is circumscribed by the rectangle determined by the lines $x = \pm\gamma_{jn}$ and $y = \pm\gamma_{j+1,n}$. A complete description of $\mathscr{A}_j$ is given by the following

THEOREM 1. *Let $0 \leq j \leq n-2$. The boundary of $\mathscr{A}_j$ is given parametrically in t by the curve*

$$x(t) = \mathscr{E}_n^{(j)}(t)$$
$$y(t) = \mathscr{E}_n^{(j+1)}(t).$$

Since $\mathscr{E}_n(t)$ is periodic with period 2 the boundary of $\mathscr{A}_j$ is parameterized over the finite interval [0, 2] and is, of course, a simple closed curve. For $j = 0$, the result is already implicit in Kolmogorov's paper of 1939 [4]. This case is formulated there as an auxiliary inequality used in the induction proof of the main result (1.2) on norm inequalities. The case $j = n - 1$ is exceptional in that $\mathscr{A}_{n-1}$ reduces to a rectangle. The contribution of the present paper lies in its methods and the cases $j = 1,..., n - 2$. In Section 2 we present certain interpolation formulas of cardinal type and use these to give a proof of Theorem 1. We derive these formulas in Section 3.

2. Some Formulas of Cardinal Type: a Proof of Theorem 1

We could define the sets $\mathscr{A}_j$ for function classes other than $\mathscr{F}_n$. For example, let B_π denote all entire functions of exponential type π which when restricted to the real axis are uniformly bounded by 1. As above, put

$$\tilde{\mathscr{A}} = \{(f(s), f'(s)) \mid f \in B_\pi, s \text{ real}\}.$$

For $\tilde{\mathscr{A}}$ we have a

Proposition. *The boundary of $\tilde{\mathscr{A}}$ is given parametrically by the curve* $(\cos \pi t, -\pi \sin \pi t)$.

This proposition is implicit in earlier work of Duffin and Schaeffer [3], and indeed follows quite easily from a formula of Pólya–Szegö [7; II, 165]. Our use of this formula demonstrates the method by which we will derive our Theorem 1.

Proof. We exploit the following formula, valid for any $f \in B_\pi$ and any t, real or complex:

$$\pi \cos \pi t f(t) - \sin \pi t f'(t) = \frac{1}{\pi} \sum_{\nu=-\infty}^{\infty} (-1)^\nu \frac{\sin^2 \pi t}{(t-\nu)^2} f(\nu)$$

$$= \sum_{\nu=-\infty}^{\infty} A_\nu f(\nu) \tag{2.1}$$

where the last equality merely serves to define the coefficients A_ν of the formula. Note that when t is real

$$\operatorname{sign} A_\nu = (-1)^\nu$$

unless t is an integer for then all but one of the A_ν vanish.

Now as in the introduction $\tilde{\mathscr{A}}$ is viewed as a convex subset of the $x - y$

plane. So $\tilde{\mathscr{A}}$ has a supporting line with normal vector (α, β), see Figure 1, and the position of this line is determined by

$$\max\{\alpha x + \beta y \mid (x, y) \in \tilde{\mathscr{A}}\}. \tag{2.2}$$

Setting

$$\alpha = \pi \cos \pi t, \qquad \beta = -\sin \pi t \tag{2.3}$$

for an appropriate t, we see that the corresponding quantity in (2.2) becomes

$$\pi \cos \pi t f(s) - \sin \pi t f'(s) \tag{2.4}$$

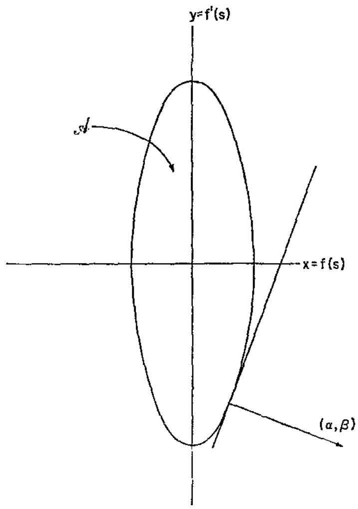

FIGURE 1

which must be maximized over all $f \in B_\pi$ and over all real s. But B_π is invariant under shifts, so we may just as well take $s = t$ in (2.4) and so recover the left hand side of (2.1). Given the alternating signs of A_ν, formula (2.1) then makes clear that (2.4) with s replaced by t is maximized when the function $f(s)$ is $\cos \pi s$; hence

$$(\cos \pi t, -\pi \sin \pi t) \in \partial\tilde{\mathscr{A}}.$$

This result persists for every t, and varying t we generate every normal direction (α, β) as seen from (2.3). Thus $(\cos \pi t, -\pi \sin \pi t)$ describes the full boundary of $\tilde{\mathscr{A}}$, as was to be shown.

After this short digression, we return to our main interest: the function class $\mathscr{F}_n$ and the corresponding sets $\mathscr{A}_j$, $j = 0,..., n-2$. Our main goal is a class of formulas analogous to (2.1). The existence and character of these formulas is the content of

THEOREM 2. *Fix n and j with $n \geqslant 4$ and $0 \leqslant j \leqslant n-2$. Also fix a real value t. Then for any $f \in \mathscr{F}_n$ we have*

$$\mathscr{E}_{n-1}^{(j)}(t+\tfrac{1}{2})f^{(j+1)}(t) - \mathscr{E}_{n-1}^{(j+1)}(t+\tfrac{1}{2})f^{(j)}(t) = \sum_{\nu=-\infty}^{\infty} A_\nu f(\nu) + \int_{-\infty}^{\infty} K(s) f^{(n)}(s)\,ds \tag{2.5}$$

where

(i) $(-1)^\nu A_\nu > 0$;

(ii) *$K(s)$ is, except for a discontinuity at t, a cardinal spline of degree $n-1$ with knots at the integers; the discontinuity at t is in $K^{(n-j-1)}$ and $K^{(n-j-2)}$;*

(iii) *for n even* (2.6)

$$(-1)^{\nu+n/2} K(s) > 0 \qquad \text{if} \quad \nu - \tfrac{1}{2} < s < \nu + \tfrac{1}{2};$$

for n odd

$$(-1)^{\nu+(n-1)/2} K(s) > 0 \qquad \text{if} \quad \nu - 1 < s < \nu;$$

(iv) *both A_ν and $K(s)$ tend exponentially to 0 as $|\nu|$ and $|s|$ tend to infinity.*

Remarks. The A_ν and $K(s)$ both depend of course on t, but we do not indicate this in the notation. Formula (2.5) is valid for every f with $f^{(n)}$ essentially bounded and $f^{(n-1)}$ absolutely continuous. The case $j=0$ and t an integer is exceptional as then the left hand side of (2.5) collapses to a multiple of $f(t)$.

The proof of Theorem 2, which is technically complicated, we defer to Section 3. Here instead we give in detail some special cases and then indicate how (2.5) and (2.6) are used to prove Theorem 1. We observe that the very *existence* of formula (2.5) with properties (2.6) is enough to establish the extremal property of $\mathscr{E}_n(s)$ given in Theorem 1.

For our first example of the type of formula contained in Theorem 2, set $n = 3$ and $j = 0$. We find that for $0 \leqslant t \leqslant 1$

$$\mathscr{E}_2(t+\tfrac{1}{2})f'(t) - \mathscr{E}_2'(t+\tfrac{1}{2})f(t) = 4(t-1)^2 f(0) - 4t^2 f(1) + \int_0^1 K_t(s) f^{(3)}(s)\,ds \tag{2.7}$$

where

$$K_t(s) = 2(t-1)^2 s^2 \qquad 0 \leqslant s \leqslant t$$
$$= 2t^2(s-1)^2 \qquad t \leqslant s \leqslant 1.$$

Note that $K_t(s) \geqslant 0$. There are formulas similar to (2.7) for other values of t; but due to the symmetries of $\mathscr{A}_0$, (2.7) is sufficient for our needs.

For $n = 3$, $j = 1$ and $0 \leqslant t \leqslant 1$ the required formula is

$$\mathscr{E}_2'(t+\tfrac{1}{2}) f''(t) - \mathscr{E}_2''(t+\tfrac{1}{2}) f'(t) = 8f(0) - 8f(1) + \int_0^1 K_t(s) f^{(3)}(s)\, ds \quad (2.8)$$

where

$$K_t(s) = 4s^2 \qquad 0 \leqslant s \leqslant t$$
$$= 4(s-1)^2 \qquad t \leqslant s \leqslant 1.$$

When $t = 0$, we infer by continuity that the coefficient of $f'(t)$ in (2.8) is -8; (2.8) thus reduces to the Taylor expansion for $f(1)$ about the origin.

The case $n = 3$ and also $n = 2$, which we omit, are exceptional in that our formulas are finite in nature. The situation changes for $n \geqslant 4$, for then we have the full force of Theorem 2 and the formulas are truly of cardinal type, involving all integers ν as nodes and kernels $K(s)$ supported on the entire real axis. The first such we encounter is for $n = 4$ and $j = 0$:

$$\mathscr{E}_3(t+\tfrac{1}{2}) f'(t) - \mathscr{E}_3'(t+\tfrac{1}{2}) f(t) = \sum_{-\infty}^{\infty} A_\nu f(\nu) + \int_{-\infty}^{\infty} K(s) f^{(4)}(s)\, ds \quad (2.9)$$

where

(i) $A_\nu = a_1(t)\, \lambda_1^{\nu}, \qquad \nu \geqslant 1$

$= a_2(t)\, \lambda_2^{\nu}, \qquad \nu \leqslant -1$

$\lambda_1 = -11 + 2(30)^{1/2} = -.045548, \qquad \lambda_2 = \lambda_1^{-1}$

(ii) for $0 \leqslant t \leqslant \frac{1}{2}$ and $\mu = \dfrac{(1-\lambda_1)^2}{1+\lambda_1}$ (2.10)

$$a_i(t) = \frac{\mu}{2}\left[\frac{\lambda_1 - 1}{\lambda_1} t^2(4t^2+3) + (-1)^i \frac{1+\lambda_1}{\lambda_1} 8t^3\right], \qquad i = 1, 2$$

$$A_0 = A_0(t) = -3(4t^2-1) + \mu t^2(4t^2+3)$$

(iii) $K(s)$ is a cubic spline with knots at the integers and at t; and

$$(-1)^\nu K(s) > 0 \qquad \text{for } \nu - \tfrac{1}{2} < s < \nu + \tfrac{1}{2}.$$

An easy calculation from (ii) shows that

$$a_i(t) > 0, \qquad i = 1, 2 \quad \text{and} \quad A_0(t) > 0;$$

hence with $\lambda_i < 0$, $i = 1, 2$, (i) implies

$$(-1)^\nu A_\nu > 0 \qquad \text{for all } \nu.$$

Concerning the sign regularity of $K(s)$ given by (iii), we make a series of remarks. From our construction of $K(s)$ in Section 3, it will be clear that $K(s)$ has simple zeros at every point $\nu + \frac{1}{2}$. Once K is constructed (2.9) emerges when we integrate by parts the remainder

$$\int_{-\infty}^{\infty} K(s) f^{(4)}(s)\, ds.$$

It follows that

$$K'''(1+) - K'''(1-) = -A_1 > 0$$

so

$$K'''(s) > 0, \qquad 1 < s < 2$$

and in particular

$$K'''(\tfrac{3}{2}) > 0.$$

Anticipating considerations of Section 3, we find three (weak) sign changes in the sequence

$$K(\tfrac{3}{2}),\ K'(\tfrac{3}{2}),\ K''(\tfrac{3}{2}),\ K'''(\tfrac{3}{2})$$

which then forces

$$K'(\tfrac{3}{2}) \geqslant 0.$$

This together with the simple zeros of $K(s)$ at $\nu + \frac{1}{2}$ yields the particular sign pattern (iii) of (2.10).

When $t = \frac{1}{2}$ (2.9) becomes a formula for $f'(\frac{1}{2})$: it is, after multiplication by -1, precisely the formula given by Schoenberg in [8] and again in [9], as is seen when (2.10) is evaluated for $t = \frac{1}{2}$. More generally the formulas of Theorem 2 reduce to formulas of C. de Boor and I. J. Schoenberg [1] when

$$j \text{ even and } t = \tfrac{1}{2}$$

or when

$$j \text{ odd and } t = 0.$$

For the case $j = 0$ and $t = \frac{1}{2}$, the formula had been established by C. A. Micchelli [5] in his 1974 lecture at the Weizmann Institute.

For $n = 4$ and $j = 1$ the formula is

$$\mathscr{E}_3'(t + \tfrac{1}{2}) f''(t) - E_3''(t + \tfrac{1}{2}) f'(t) = \sum_{-\infty}^{\infty} A_\nu f(\nu) + \int_{-\infty}^{\infty} K(s) f^{(4)}(s)\, ds \quad (2.9)_\prime$$

with A_ν and $K(s)$ given just as in (2.10) except that (ii) is replaced

$$a_i(t) = 3\mu \left[\frac{\lambda_1 - 1}{\lambda_1} (4t^2 + 1) + (-1)^i \frac{1 + \lambda_1}{\lambda_1} 4t\right], \qquad i = 1, 2 \quad (2.10)_\prime$$

$$A_0 = A_0(t) = 6\mu(4t^2 + 1), \qquad 0 \leqslant t \leqslant \tfrac{1}{2}.$$

Clearly $A_0(t) > 0$ as are $a_i(t)$, $0 \leqslant t \leqslant \frac{1}{2}$. Thus the A_ν of $(2.9)_\prime$ have the desired sign pattern $(-1)^\nu A_\nu > 0$.

And as a last example for $n = 4$ and $j = 2$ we have

$$\mathscr{E}_3''(t + \tfrac{1}{2}) f'''(t) - \mathscr{E}_3'''(t + \tfrac{1}{2}) f''(t) = \sum_{-\infty}^{\infty} A_\nu f(\nu) + \int_{-\infty}^{\infty} K(s) f^{(4)}(s)\, ds \quad (2.9)_{\prime\prime}$$

where now (ii) of (2.10) is replaced by

$$a_1(t) = a_2(t) = -24\mu \frac{1 - \lambda_1}{\lambda_1}$$

$$A_0 = 48\mu. \qquad (2.10)_{\prime\prime}$$

Having thus concluded our examples of some of the formulas contained in Theorem 2, we now use the general formula to prove Theorem 1. The argument is along lines very like those used above to derive the Proposition concerning $\tilde{\mathscr{A}}$ from the Pólya–Szegö formula (2.1).

Proof of Theorem 1. Each $\mathscr{A}_j$ is a convex set and so can be completely described in terms of its lines of support. Just as in the proof of the Proposition, determining the position of the supporting lines in a given direction with normal (α, β) amounts to maximizing

$$\mathscr{E}_{n-1}^{(j)}(t + \tfrac{1}{2}) f^{(j+1)}(t) - \mathscr{E}_{n-1}^{(j+1)}(t + \tfrac{1}{2}) f^{(j)}(t) \qquad (2.11)$$

over all $f \in \mathscr{F}_n$. For $f \in \mathscr{F}_n$ we evaluate (2.11) via (2.5) as

$$\sum_{\nu=-\infty}^{\infty} A_\nu f(\nu) + \int_{-\infty}^{\infty} K(s) f^{(n)}(s)\, ds \leqslant \sum_{\nu=-\infty}^{\infty} |A_\nu| + \gamma_{nn} \int_{-\infty}^{\infty} |K(s)|\, ds \qquad (2.12)$$

where the inequality follows from conditions (1.1) defining the class $\mathscr{F}_n$.

Clearly equality occurs in (2.12) if and only if f satisfies both

$$f(\nu) = \text{signum } A_\nu = (-1)^\nu$$

and

$$f^{(n)}(s) = \gamma_{nn} \text{ signum } K(s) \text{ a.e.}$$

But by comparing (1.3) with (2.6), we see that these two conditions are satisfied by the Euler spline $\mathscr{E}_n$, and, in fact, they characterize it. So (2.11) is maximized when $f(s) \equiv \mathscr{E}_n(s)$. This implies that the pair $(\mathscr{E}_n^{(j)}(t), \mathscr{E}_n^{(j+1)}(t))$ is on the boundary of $\mathscr{A}_j$, and as we vary t we generate the entire boundary. We mention in passing that the ratio

$$\frac{\alpha}{\beta} = -\frac{\mathscr{E}_{n-1}^{(j+1)}(t+\frac{1}{2})}{\mathscr{E}_{n-1}^{(j)}(t+\frac{1}{2})}$$

takes on every extended real value and so maximizing (2.11) gives supporting lines in every possible direction. ∎

From the uniqueness comments just made in the proof of Theorem 1, we obtain the following remarkable property of the Euler spline:

COROLLARY. *Assume $n \geqslant 3$ and $f \in \mathscr{F}_n$. The equations*

$$f^{(j)}(t) = \mathscr{E}_n^{(j)}(t)$$
$$f^{(j+1)}(t) = \mathscr{E}_n^{(j+1)}(t)$$

can hold simultaneously at some point t only if

$$f(s) \equiv \mathscr{E}_n(s)$$

for all real s. If $j = 0$ we exclude from our assertion any integral value of t.

In other words, the pair $(f^{(j)}(t), f^{(j+1)}(t))$ is *always in the interior* of $\mathscr{A}_j$ unless f is the Euler spline $\mathscr{E}_n$ in which case the pair is *always on the boundary* of $\mathscr{A}_j$.

3. A Construction of the Formulas of Theorem 2

We will carry out the construction for n even; for the case of n odd, small variations are necessary. Our main task is to construct $K(x)$ with the properties given by (2.6). The formula (2.5) then emerges easily by integration by parts.

There are two main tools involved in the construction, tools from the theory of cardinal spline functions. Our references for this material are [9] of Schoenberg and [1] of de Boor–Schoenberg to which we refer the reader for details; also to [5] and [6] where C. A. Micchelli has developed some of these methods as early as January 1974 to provide an "optimal estimator" for $f'(t)$.

The eigensplines. These are cardinal splines S satisfying the functional equation

$$S(x+1) = \lambda S(x).$$

The number λ is called the eigenvalue. We need two classes of such eigensplines, those vanishing at the integers ν and also those which vanish at the points $\nu + \frac{1}{2}$; in both cases the knots are to be *at the integers.*

When $n = 2m$ the degree of K is $2m - 1$ and according to (2.6) K must have sign changes at $\nu + \frac{1}{2}$ for every integer ν. We find in [9] $2m - 1$ eigenvalues μ_ν

$$\mu_1 < \cdots < \mu_{m-1} < \mu_m = -1 < \mu_{m+1} < \cdots < \mu_{2m-1} < 0$$

and corresponding eigensplines

$$S_i(x), \qquad i = 1,..., 2m - 1 \tag{3.1}$$

of degree $2m - 1$ satisfying

$$\begin{aligned} S_i(\tfrac{1}{2}) &= 0 \\ S_i(x+1) &= \mu_i S_i(x) \qquad \text{for all } x. \end{aligned} \tag{3.2}$$

We note that for $i = m + 1,..., 2m - 1$

$$\lim_{x\to\infty} S_i(x) = 0;$$

and for this reason these $S_i(x)$, $(i = m + 1,..., 2m - 1)$ are sometimes called the "decreasing" eigensplines. Among the eigensplines (3.1), $S_m(x)$ is the only one which is bounded.

When $n = 2m + 1$, we again find [9] $2m - 1$ eigenvalues

$$\lambda_1 < \cdots < \lambda_{m-1} < \lambda_m = -1 < \lambda_{m+1} < \cdots < \lambda_{2m-1} < 0$$

and corresponding eigensplines

$$\tilde{S}_i(x), \qquad i = 1,..., 2m - 1 \tag{3.1},$$

of degree $2m$ satisfying

$$\tilde{S}_i(0) = 0$$
$$\tilde{S}_i(x+1) = \lambda_i \tilde{S}_i(x) \qquad \text{for all } x. \tag{3.2}$$

The Budan–Fourier Theorem for splines. For a given spline function f, we let $Z_f(a, b)$ denote the number of zeros of f, counting multiplicity, on the open interval (a, b). If f is of degree n, $f^{(n)}$ is piecewise constant and $Z_{f^{(n)}}(a, b)$ is defined as the number of strong sign changes on (a, b); thus an interval where $f^{(n)}$ vanishes identically is ignored. In addition

$$S^-(f(a),..., f^{(n)}(a))$$

denotes the number of sign changes in the sequence $f(a),..., f^{(n)}(a)$ where zeros are ignored. Similarly,

$$S^+(f(a),..., f^{(n)}(a))$$

counts the sign changes with zeros taken positive or negative so as to maximize the count. With these notations we state the useful

THEOREM. *Assume that the spline f is of precise degree n and has a finite number of simple knots in (a, b). Then*

$$Z_f(a, b) \leqslant Z_{f^{(n)}}(a, b) + S^-(f(a),..., f^{(n)}(a+)) - S^+(f(b),..., f^{(n)}(b-)). \tag{3.3}$$

There are many references to this result; perhaps the most accessible for the present purposes is [1] or [6].

The use of the Budan–Fourier Theorem in the presence of eigensplines is very much facilitated by the following proposition which plays a very important role in our construction.

PROPOSITION. (1) *The eigensplines $S_i(x)$ of* (3.1) *satisfy for every integer ν*

$$S^-(S_i(\nu + \tfrac{1}{2}),..., S_i^{(2m-1)}(\nu + \tfrac{1}{2})) = i - 1$$
$$S^+(S_i(\nu + \tfrac{1}{2}),..., S_i^{(2m-1)}(\nu + \tfrac{1}{2})) = i. \tag{3.4}$$

(2) *The eigensplines $\tilde{S}_i(x)$ of* (3.1), *satisfy for every integer ν*

$$S^-(\tilde{S}_i(\nu),..., \tilde{S}_i^{(2m)}(\nu+)) = i$$
$$S^+(\tilde{S}_i(\nu),..., \tilde{S}_i^{(2m)}(\nu-)) = i. \tag{3.4'}$$

The proposition appears in [1] and also [6]; it is proved on the basis of the Gantmacher–Krein Theorem on oscillation matrices.

Determining the kernel $K(x)$. Set $n = 2m$, $m \geqslant 2$. Fix j and t as in Theorem 2. For simplicity we assume $0 \leqslant t \leqslant \frac{1}{2}$; clearly this represents no essential restriction. Also if $j = 0$ we exclude $t = 0$ as indicated in the remarks following the statement of Theorem 2. Put

$$K(x) = K_1(x) = \sum_{i=1}^{m-1} a_i S_i(x) + a x_+^{2m-1} + b(x-t)_+^{2m-1-j} + c(x-t)_+^{2m-2-j}, \qquad x \leqslant 1$$

$$= K_2(x) = \sum_{i=m+1}^{2m-1} a_i S_i(x), \qquad x \geqslant t$$

for an appropriate choice of the $2m + 1$ parameters

$$\{a_1, \ldots, a_{m-1}, a_{m+1}, \ldots, a_{2m-1}, a, b, c\}$$

to be determined presently. Note that in (3.5) each $S_i(x)$, $i = 1, \ldots, m-1$, is extended from the interval $(-1, 0)$ to $(-1, 1)$ without a knot at 0; instead the term $a x_+^{2m+1}$ provides the knot at 0 for $K(x)$. To check that $K(x)$ is well defined by (3.5) as a single valued function, both definitions of $K(x)$ given by $K_1(x)$ and $K_2(x)$ must agree on the overlap $(t, 1)$. So when restricted to the interval $(t, 1)$, $K_1(x)$ and $K_2(x)$ must be *identically the same polynomial.* Equivalently

$$K_1^{(l)}(\tilde{t}) = K_2^{(l)}(\tilde{t}) \qquad l = 0, \ldots, 2m-1$$

for any fixed $\tilde{t}$ with $t < \tilde{t} < 1$. After a little rearrangement, these conditions yield a linear system of $2m$ equations in the $2m$ unknowns

$$\{a_1, \ldots, a_{m-1}, a_{m+1}, \ldots, a_{2m-1}, b, c\}$$

with a right hand side given by the term $a x_+^{2m-1}$ evaluated at $x = \tilde{t}$.

To determine a solution of the above linear system, consider first the homogeneous system obtained by setting $a = 0$. Suppose there were a nontrivial solution. Then the result is a $K(x)$ defined by (3.5) on the entire axis but with *no knot* at 0 since $a = 0$. Now it is easily seen that the sum

$$K(x) = \sum_{i=m+1}^{2m-1} a_i S_i(x), \qquad x \geqslant t \tag{3.6}$$

is nontrivial. In addition, the functional equations (3.2) and the ordering of the eigenvalues μ_i together imply that for large values of the argument x the sum (3.6) is dominated by $S_{m+1}(x)$. Thus from (3.4) with ν large we have

$$S^+(K(\nu + \tfrac{1}{2}), K'(\nu + \tfrac{1}{2}),..., K^{(2m-1)}(\nu + \tfrac{1}{2})) \leqslant m + 1. \tag{3.7}$$

Similarly

$$S^-(K(-\nu + \tfrac{1}{2}), K'(-\nu + \tfrac{1}{2}),..., K^{(2m-1)}(-\nu + \tfrac{1}{2})) \leqslant m - 2. \tag{3.8}$$

Now using these two estimates, we apply on each of the intervals $(-\nu + \frac{1}{2}, t)$ and $(t, \nu + \frac{1}{2})$ the Budan–Fourier Theorem (3.3) to $K(x)$ given by (3.5) with $a = 0$. When the resulting two inequalities are added together, we obtain

$$\begin{aligned} 2\nu - 1 \leqslant{} & 2\nu - 1 + S^-(K(t+), K'(t+),..., K^{(2m-1)}(t+)) \\ & - S^+(K(t-), K'(t-),..., K^{(2m-1)}(t-)) + (m-2) - (m+1) \\ \leqslant{} & 2\nu - 1 + 2 + (m-2) - (m+1) = 2\nu - 2. \end{aligned} \tag{3.9}$$

The second inequality of (3.9) follows because the two sequences

$$K(t+), K'(t+),..., K^{(2m-1)}(t+)$$
$$K(t-), K'(t-),..., K^{(2m-1)}(t-)$$

can differ (by (3.5)) only in two consecutive entries; hence the corresponding difference in (3.9) is at most 2. Now (3.9) is a contradiction, implying that the homogeneous system has only the trivial solution.

Now set $a = 1$ and so obtain a unique $K(x)$ defined by (3.5). To this function $K(x)$ we again apply the above arguments leading to (3.9) but now with the one change that 0 is a knot. The result, valid for all large integers ν, is

$$2\nu - 1 \leqslant 2\nu + 2 + (m-2) - (m+1) = 2\nu - 1. \tag{3.10}$$

So we must have equality in (3.10) which forces equality in (3.7) and (3.8). From these equalities we can easily derive all the properties asserted for $K(x)$ in Theorem 2.

Properties of $K(x)$. (ii) of (2.6) is clear from (3.5), as is the exponential decay of $K(x)$. From (3.2) and (3.5), $K(\nu + \frac{1}{2}) = 0$ for all integers ν; that these zeros are simple, and that $K(x)$ has no other zeros, follows from (3.10). Thus $K(x)$ changes sign at each point $\nu + \frac{1}{2}$.

Again from (3.10), $K^{(2m-1)}(x)$ must change sign across every integer and when $j \geqslant 1$ these are clearly the only sign changes of $K^{(2m-1)}(x)$. For $j = 0$

there is a possible sign change at t, but we will eliminate this possibility shortly. Formula (2.5) emerges by integrating by parts

$$\int_{-\infty}^{\infty} K(x) f^{(2m)}(x)\, dx.$$

Thus the A_ν of (2.5) are given by

$$A_\nu = -(K^{(2m-1)}(\nu+) - K^{(2m-1)}(\nu-)). \tag{3.11}$$

So

$$A_\nu \text{ strictly alternates in sign} \tag{3.12}$$

and we normalize by

$$\operatorname{sign} A_0 > 0.$$

This normalization implies

$$K^{(2m-1)}(x) > 0 \quad \text{for} \quad -1 < x < 0$$
$$K^{(2m-1)}(x) > 0 \quad \text{for} \quad -2\nu - 1 < x < -2\nu;$$

in particular

$$K^{(2m-1)}(-2\nu - \tfrac{1}{2}) > 0. \tag{3.13}$$

Now equality in (3.8) combined with (3.12) yields

$$\operatorname{sign} K'(-2\nu - \tfrac{1}{2}) = (-1)^{m-2} = (-1)^m. \tag{3.14}$$

So

$$(-1)^m K(x) > 0 \quad \text{for} \quad -2\nu - \tfrac{1}{2} < x < -2\nu + \tfrac{1}{2}$$

and due to the simplicity of the zeros of $K(x)$ we find

$$(-1)^{m+\nu} K(x) > 0 \quad \text{for} \quad \nu - \tfrac{1}{2} < x < \nu + \tfrac{1}{2} \tag{3.15}$$

valid for all ν. This establishes (iii) of (2.6).

Concerning the case $j = 0$, we see from (3.5) that K has a double knot at t; this allows a possible change of sign in $K^{(2m-1)}$ at t, and we must eliminate this possibility in order to preserve (3.12). Given the sign changes of $K^{(2m-1)}$ at every integer, a sign change at t would entail

$$K^{(2m-1)}(2\nu - \tfrac{1}{2}) < 0$$

for large positive ν. Following the same line of reasoning which resulted in (3.15), we would arrive at

$$(-1)^{m+\nu} K(x) < 0 \qquad \text{for} \quad \nu - \tfrac{1}{2} < x < \nu + \tfrac{1}{2}.$$

This contradicts (3.15); hence there is no sign change of $K^{(2m-1)}$ at t.

Thus we have established formula (2.5) with a right hand side described by (2.6). From (3.5) and the integration by parts, it is clear that the left hand side of our formula is of the form

$$\alpha f^{(j)}(t) + \beta f^{(j+1)}(t).$$

We have yet to determine α and β, or more precisely the ratio β/α, as our formula is determined only up to a multiplicative constant.

Recall the sets $\mathscr{A}_j$ of Section 1. For every s

$$(\mathscr{E}_n^{(j)}(s), \mathscr{E}_n^{(j+1)}(s)) \in \mathscr{A}_j ;$$

and in fact on the basis of all the properties (2.6) of formula (2.5) and the corresponding properties (1.3) of $\mathscr{E}_n(s)$, we can already conclude as in Section 2 that

$$(\mathscr{E}_n^{(j)}(s), \mathscr{E}_n^{(j+1)}(s)) \in \partial\mathscr{A}_j .$$

With

$$x = \mathscr{E}_n^{(j)}(s), \qquad y = \mathscr{E}_n^{(j+1)}(s)$$

we find

$$\frac{dy}{dx} = \frac{dy}{ds}\Big/\frac{dx}{ds} = \frac{\mathscr{E}_n^{(j+2)}(s)}{\mathscr{E}_n^{(j+1)}(s)} = \frac{\mathscr{E}_{n-1}^{(j+1)}(s + \frac{1}{2})}{\mathscr{E}_{n-1}^{(j)}(s + \frac{1}{2})} .$$

And from Figure 1 it is clear that

$$-\frac{\alpha}{\beta} = \frac{dy}{dx}\Big|_{s=t}$$

So we have $\beta = \mathscr{E}_{n-1}^{(j)}(t + \frac{1}{2})$ and $\alpha = -\mathscr{E}_{n-1}^{(j+1)}(t + \frac{1}{2})$.

The odd case $n = 2m + 1$ is settled in exactly the same way with the even degree eigensplines $\tilde{S}_i(x)$ given by (3.1), and (3.2), replacing the $S_i(x)$. One then argues on the integer points $x = \nu$, as indicated by (3.4), .

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